An ABC-Problem for Location and Consensus Functions on Graphs

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8 June 2015

Econometric Institute Report EI 2015-16

Abstract

A location problem can often be phrased as a consensus problem or a voting problem. We use these three perspectives, namely location, consensus and voting to initiate the study of several questions. The median function *Med* is a location/consensus function on a connected graph G that has the finite sequences of vertices of G as input. For each such sequence π, *Med* returns the set of vertices that minimize the distance sum to the elements of π. The median function satisfies three intuitively clear axioms: (A) Anonymity, (B) Betweenness and (C) Consistency. In [24] it was shown that on median graphs these three axioms actually characterize *Med*. This result raises a number of questions:

(i) On what other classes of graphs is *Med* characterized by (A), (B) and (C)?

(ii) If some class of graphs has other ABC-functions besides *Med*, then determine additional axioms that are needed to characterize *Med*.

(iii) In the latter case, can we find characterizations of other functions that satisfy (A), (B) and (C)?

We call these questions, and related questions, the ABC-Problem for location/consensus functions on graphs. In this paper we present first results. For the first question we use consensus terminology. We construct a non-trivial class different from the median graphs, on which the median function is the unique "ABC-function". For the second and third question voting terminology is most apt for our approach. On K_n with $n > 2$ we construct various non-trivial ABC-voting procedures. For some nice families, we present a full axiomatic characterization. We also construct an infinite family of ABC-functions on K_3.

Keywords: median function; location function; consensus function; voting; consensus axiom; ABC-problem

Mathematics Subject Classification 2010: 05C90, 90B80, 05C75

1 Introducing the ABC-Problem

The notions of location and consensus can often be considered the same formally. To illustrate this suppose G is a finite connected graph and we have a set $\{1, \ldots, k\}$ of clients (voters). Each client i selects a most suitable, or preferred, location x_i in V, and it is the task of a location (consensus) function to return those vertices that best satisfy various constraints and properties deemed appropriate for the particular problem at hand. In location problems the constraints are usually in the form of optimizing certain criteria. In consensus problems one usually requires certain simple and acceptable rules or axioms that make the voting a reasonable and rational procedure. Both problems can also be phrased as a voting procedure. In this paper we use these different perspectives, and each time choose the one that is closest to intuition for the particular problem being addressed.

One of the early papers on location problems is the classical paper of Witzgall in 1965 [31]. Since then hundreds of papers have been written about location problems on graphs using the geodesic metric: for example see the reference lists in [25, 26, 27, 28, 29]. The earliest paper on the axiomatic study of consensus is the classical paper of Arrow in 1951 [1]. This was the beginning of a fruitful and rich area of research, see for example [2, 3, 5, 6, 7, 27]. Holzman [9] was the first to study a location function as a consensus problem, that is, finding axiomatic characterizations of location functions, thus combining the areas of location and consensus. For some recent work in this area, see [12, 13, 15, 17, 18, 30].

A typical location problem is finding a median of a set of clients on a connected graph G, where a median is a vertex that minimizes the distance sum to the clients. It is usually modeled as a consensus function *Med* that returns the set of all medians. In 1996 Vohra [30] characterized the median function axiomatically on tree networks (the continuous version of a tree, where internal vertices of edges are also possible locations). In 1998 McMorris, Mulder and Roberts [16] handled the discrete case. They were able to characterize *Med* on cube-free median graphs using the three simple axioms Anonymity (A), Betweenness (B) and Consistency (C). For the general case of arbitrary median graphs they needed an extra axiom, but it was not clear then whether this extra axiom was necessary. Median graphs are a natural common generalization of trees and hypercubes [22]. They are defined by the property that, for any three vertices u, v, w there is a unique vertex that lies simultaneously on a shortest path between each pair of u, v, w. At first sight these graphs seem to be quite esoteric, but in [10] a one-to-one correspondence was established between a special subclass of the median graphs and the connected triangle-free graphs.

Recently Mulder and Novick [24] settled the unclarity in [16]: the three basic axioms (A), (B) and (C) actually characterize the median function on any median graph. Calling a consensus function on a connected graph satisfying (A),(B), and (C) an ABC-function, the Mulder & Novick result motivates the following questions and problems.

(i) Determine the classes of graphs, on which *Med* is the unique ABC-function.

(ii) If $\mathcal{G}$ is a class of graphs having other ABC-functions besides *Med*, then determine additional axioms that are needed to characterize *Med* on $\mathcal{G}$.

(iii) On such a class of graphs $\mathcal{G}$, study the other ABC-functions, and, if possible, characterize these axiomatically.

This set of questions, and related questions on ABC-functions, is the *ABC-Problem* for location and consensus functions on graphs.

The aim of this paper is to provide some first answers to these questions. First we set the stage in Section 2. In Section 3 we formally define our three basic axioms (A), (B) and (C), and give some additional relevant axioms. In Section 4 we present a non-trivial example of a class on which the median function is the unique ABC-function. Here we use the location/consensus perspective. From Section 5 onwards we use voting terminology. The complete graphs with $n \geq 3$ vertices are the point of focus here. On K_n any consensus function is basically a voting procedure on n alternatives. There exists a vast literature on voting. Our perspective here is slightly different, because we consider only voting procedures that satisfy (A), (B) and (C). But these still make eminent sense form the point of view of voting theory. For some we have nice characterizations, for others we have first results. What all this shows is that, on K_n with $n \geq 3$, ABC-functions are abundant. Moreover, we even have an infinite family on K_3. Of course this all amounts to only first steps in the study of the ABC-Problem for location and consensus functions on graphs.

2 Setting the stage

The process of finding an optimal location or reaching consensus about a certain issue can often be phrased as a voting procedure. For many types of voting procedures it is advantageous to consider the population of voters to be ordered. Hence we number the voters. We represent the alternatives on which the voters may cast their votes as vertices in a graph. Usually we equate the voter with the candidate on which the voter casts her/his vote, that is, we list the voters/votes as a sequence of vertices in the graph, and we call such a sequence a profile.

The voting process is represented by a consensus function that assigns to each profile a nonempty set of vertices of the graph. A decent and reasonable voting procedure follows some rules. These rules can be phrased as axioms for the associated consensus function. To guarantee decency of the voting procedure we want these axioms to be appealing and as simple as possible, so they are convincing for the voters. Let us formalize this.

Let $G = (V, E)$ be a finite connected graph with vertex set V. A *profile* of *length* k on V is a nonempty sequence $\pi = (x_1, x_2, \ \dots \ , x_k)$. Note that vertices may occur more than once in a profile. We denote the length of π by $k = |\pi|$. We call $x_1, \dots x_k$ the *elements* of π. A *vertex* of π is a vertex that occurs as an element in π. The *carrier set* $\{\pi\}$ of π is the set of vertices that occur in π. So, if a vertex occurs more than once as an element in π, then we have $|\{\pi\}| < |\pi|$. A *subprofile* of π is just a subsequence of π. For convenience we allow a subprofile to be empty. We denote by V^* the set of all profiles of finite length on V, and by $2^V - \emptyset$ the family of all nonempty subsets of V. A *consensus function* on a graph $G = (V, E)$ is a function $L : V^* \to 2^V - \emptyset$. For convenience, we will write $L(x_1, x_2, \ \dots \ , x_k)$ instead of $L((x_1, x_2, \ \dots \ , x_k))$, for any profile $\pi = (x_1, x_2, \ \dots \ , x_k)$. One of the objectives in the theory of consensus functions is the axiomatic characterization of certain types of consensus functions.

The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u, v-path or u, v-geodesic. Let $\pi = (x_1, x_2, \ \dots \ , x_k)$ be a profile on V. For a vertex v of G, we write $D(v, \pi) = \sum_{i=1}^{k} d(v, x_i)$. A vertex x minimizing this distance sum is called a *median* of π. The *median function Med* on a graph $G = (V, E)$ is the consensus function given by

$$Med(\pi) = \{ \ v \mid v \text{ is a median of } \pi \ \}.$$

Below we present some basic axioms that are satisfied by the median function.

Let $\pi = (x_1, x_2, \ \dots \ , x_k)$ be a profile of length k. A vertex with highest occurrence in π is called a *plurality vertex* of π. The *plurality set* $Pl(\pi)$ of π is the set of plurality vertices of π. Note that the plurality set of π is always nonempty. Clearly Pl can be considered as a consensus function. A vertex that occurs at least $\frac{1}{2}k$ times in π is called a *majority vertex*. We denote the set of majority vertices of π by M_π. Trivially, a majority vertex is also a plurality vertex. Note that π may contain at most two majority vertices. If π contains two majority vertices, say x and y, then π is an even profile of length $k = 2\ell$ that contains x as well as y exactly ℓ times. We call such a profile a *tie profile* on x, y. In all other cases, if π contains a single majority vertex, then it is unique, and we call such a profile a *majority profile*. Note that in this case, even if the majority vertex occurs exactly $\frac{1}{2}k$ times, it occurs more often than any other vertex. Moreover, if the unique majority vertex occurs exactly half of the times, then π must contain at least three distinct vertices. Trivially, any profile with carrier set a singleton is a majority profile. Such a profile is called a *constant profile* on x, where x is the vertex in the profile. If π contains every vertex at most once, then we call π a *single-occurrence profile*. This property is equivalent with π satisfying the equality $|\pi| = |\{\pi\}|$. The *full profile* is the single-occurrence profile, in which each vertex in V occurs once. A *uniform profile* is a profile in which the vertices that occur in π all occur the same number of times. Note that constant profiles and tie profiles are uniform, by definition. The concatenation of m copies of a profile π is written as π^m. The elements of a uniform profile π can be reordered such that the result can be written as ρ^m, for some single-occurrence profile ρ, where m is the number of occurrences of each vertex in π.

Here are two simple but handy lemmata, which are both obvious.

Lemma 1 *Let ρ and σ be tie or majority profiles with $M_\rho \cap M_\sigma \neq \emptyset$. Then their concatenation $\pi = \rho\sigma$ is a tie profile or a majority profile with $M_\pi = M_\rho \cap M_\sigma$.*

Lemma 2 *Let π and ρ be profiles. Then $Pl(\pi) \cap Pl(\rho) \neq \emptyset$ implies that $Pl(\pi\rho) = Pl(\pi) \cap Pl(\rho)$.*

The *interval* $I_G(u, v)$ between u and v in G is the set of vertices on the geodesics between u and v, that is,

$$I_G(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\}.$$

If no confusion arises, we write $I = I_G$. The function $I : V \times V \to 2^V - \emptyset$ is the *interval function* on G. A first extensive study of I can be found in [20]. A *median graph* is a graph G for which $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$, for any three vertices u, v, w of G. Trees and hypercubes are prime examples of median graphs. Other examples are the grid graphs, and the $C_{(4)}$-trees in [8]. In a sense median graphs are the natural common generalization of trees and hypercubes, see [22]. For a first extensive study of median graphs see [20]. Since then a rich structure theory has been developed, see e.g. [21]. One characterization of median graphs is that they are precisely the graphs in which $|Med(\pi)| = 1$, for any profile π of length 3.

3 The axioms

In this section we list a number of simple and appealing consensus axioms. As voting rules these all show some aspect of 'decency' in the voting procedure. Throughout let L be a consensus function on the finite connected graph $G = (V, E)$.

Anonymity (A) : For any profile $\pi = (x_1, x_2, \ldots, x_k)$ on V and any permutation σ of $\{1, 2, \ldots, k\}$, we have $L(\pi) = L(\pi^\sigma)$, where $\pi^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)})$.

The consensus function does not distinguish between voters. Be aware that (A) is not a trivial axiom: see [11] and [4], for two cases where anonymity has deep non-trivial repercussions.

Betweenness (B)**:** $L(u, v) = I(u, v)$, for all $u, v \in V$

Any vertex between two votes is equally valued.

Consistency (C)**:**

If $L(\pi) \cap L(\rho) \neq \emptyset$ for profiles π and ρ, then $L(\pi, \rho) = L(\pi) \cap L(\rho)$.

A decent voting procedure should satisfy consistency: if two voter populations agree on an output, then the aggregation of the two voter sets should agree on the same output. The way to aggregate is that we first get the votes from the one population and then from the other. Therefore, concatenation of the two sequences is how we aggregate.

The focus in this paper is on consensus functions that satisfy (A), (B) and (C). We call such a function an *ABC-function*. It is probably part of folklore that the median function *Med* satisfies these axioms on any finite metric space, in any case, see [16]. So on any connected graph there is at least one ABC-function. We have an important class of graphs, viz. the median graphs, on which the median function is actually characterized by these three axioms, see [24]. The proof of this result leans heavily on the above mentioned structure theory of median graphs. This characterization is the motivation for what we called in the Introduction an *ABC-Problem* for location and consensus functions on graphs.

We present some more axioms that instill decency and fairness into the voting process, and that are found to be useful for some characterizations.

Faithfulness (F)**:** $L(x) = \{x\}$, for all $x \in V$.

If there is only one voter then this vote is decisive. Note that (B) together with (C) implies (F): $L(x) \cap L(x) \neq \emptyset$, so $\{x\} = I(x, x) = L(x, x) = L(x) \cap L(x) = L(x)$.

Unanimity (U)**:** $L(\pi) = \{x\}$, for all constant profiles π on $x \in X$.

Clearly, (U) implies (F). And (F) together with (C) implies (U).

Plurality $(Plur)$**:** $L(\pi)$ is contained in the plurality set of π, for all profiles π.

This is another axiom that instills a kind of fairness on the voting process. A weaker version of $(Plur)$ is the following axiom.

Single Plurality (Pl_1)**:** If π has a single plurality vertex x then $L(\pi) = \{x\}$.

Below we will see that, assuming (A), (B) and (C), the axioms $(Plur)$ and (Pl_1) are equivalent on complete graphs.

Majority (Maj)**:** If $M_\pi \neq \emptyset$ then $L(\pi) = M_\pi$.

We will see that (A), (B) and (C) together imply (Maj) on complete graphs. So in this case our three favorite axioms guarantee a minimum amount of fairness of the voting procedure.

Fullness $(Full)$**:** If π is the full profile, that is, the profile containing every vertex exactly once, then $L(\pi) = V$.

Uniformity (Uni)**:** If π is a single-occurrence profile, then $L(\pi) = \{\pi\}$.

Note that (Uni) together with (A) and (C) give $L(\pi) = \{\pi\}$, for any uniform profile π. This axiom catches also fairness in some way. Trivially, it implies (F) and $(Full)$.

Support $(Supp)$**:** $L(\pi) \subseteq \{\pi\}$, for any profile π.

An alternative that is not voted for will not appear in the output.

The next proposition shows that (A), (B) and (C) already fix the output of an ABC-function L for a many simple profiles on complete graphs.

Proposition 3 *Let L be an ABC-function on K_n. Then the following holds:*
(i) $L(\pi) = \{x\}$ for each constant profile π on x, that is, L satisfies (F) and (U).
(ii) $L(\pi) = \{x, y\}$ for each tie-profile π on distinct vertices x and y.
(iii) $L(\pi) = \{x\}$, for any majority profile π with majority vertex x.

Proof. (i): We know this already for any connected graph. (ii): This is just (B), and the application of (C), for complete graphs. For (iii), let $V = \{x, y_1, \ldots, y_{n-1}\}$ and, using (A), write the majority profile with majority vertex x as the concatenation of profiles of the following types: (x, y_j) and (x). Note that either there are at least two distinct y_j-s, or there is only one $y = y_j$, but then there are more x-s than y-s. By (B) and (C), we are done. □ □ □

The following corollary is an immediate consequence.

Corollary 4 *Let L be an ABC-function on K_n. Then $L(\pi) = M_\pi = Med(\pi)$, for any profile π on K_n with $|\{\pi\}| \leq 2$.*

The median function on K_n satisfies all of the above axioms. Note that on complete graphs $Med(\pi)$ is the plurality set of π.

4 A class of graphs with a unique ABC-function

In this section we address the question of whether there are other classes of graphs besides the median graphs that admit only one ABC-function. We exhibit a non-trivial class. Of course the question remains to determine other (or all) classes that have a unique ABC-function, viz. the median function.

As observed in the introduction, our story about voting on discrete structures begins with the 1998-paper [16] of McMorris, Mulder & Roberts (inspired by the 1996-paper [30] of Vohra). There it was proved that on cube-free median graphs the median function is the only ABC-function, where a graph is *cube-free* if it does not contain the 3-cube Q_3 as an induced subgraph. The following property played a key role in the proof of this result.

Intersecting-Intervals Property: For any even profile π of length $k = 2m$ on G, there exists a permutation of π that results in a profile $(y_1, y_2, \ldots, y_{2m})$ such that

$$\bigcap_{i=1}^{m} I(y_{2i-1}, y_{2i}) \neq \emptyset.$$

Let G be a graph with this property. Since a profile of length 2 is even, it follows that any interval in G is non-empty, so G is necessarily connected.

In [16] it was proved that cube-free median graphs have this property. We use this result below. As clarification of the intersecting-intervals property see Figure 1. The graph on the right is a 3-cube with a profile of length four. Any pairing of the elements produces two intervals that have an empty intersection. In the graph on the left, which is a cube-free median graph, there exists a pairing such that the intervals intersect, but the two other pairings produce non-intersecting intervals.

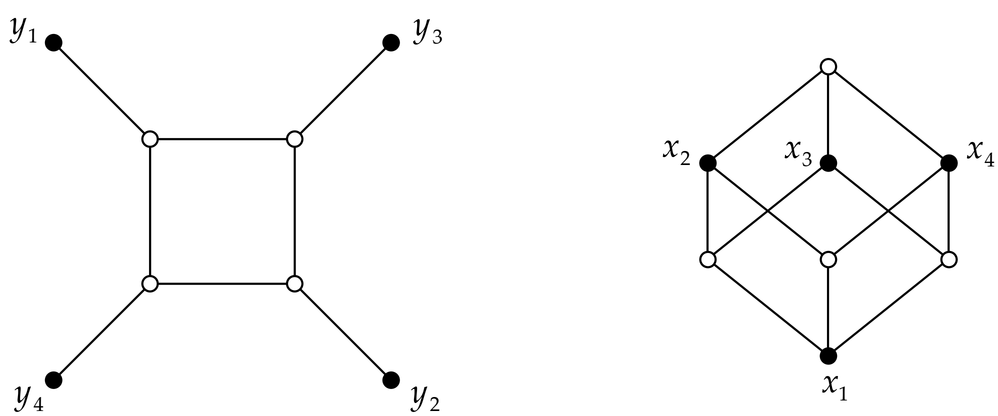

Figure 1: Graphs with a profile of length 4

For graphs having this property it is easy to prove that the median function is the unique ABC-function.

Theorem 5 *Let G be a graph having the intersecting-intervals property, and let L be a consensus function on G. Then $L = Med$ if and only if L satisfies (A), (B) and (C).*

Proof. If $L = Med$, then obviously L satisfies (A), (B) and (C).

Let L be a consensus function satisfying (A), (B) and (C). First let π be an even profile on G. Then there exists a permutation of π that results in a profile $(y_1, y_2, \ldots, y_{2m})$ such that

$$\bigcap_{i=1}^{m} I(y_{2i-1}, y_{2i}) \neq \emptyset.$$

By (A), (B) and (C) we have

$$Med(\pi) = Med(y_1, y_2, \ldots, y_{2m}) = \bigcap_{i=1}^{m} Med(y_{2i-1}, y_{2i}) =$$

$$\bigcap_{i=1}^{m} I(y_{2i-1}, y_{2i}) = \bigcap_{i=1}^{m} L(y_{2i-1}, y_{2i}) = L(y_1, y_2, \ldots, y_{2m}) = L(\pi).$$

Next, let π be an odd profile. Then $\pi\pi$ is an even profile, and, by (C), we have $L(\pi) = L\pi) \cap L(\pi) = L(\pi\pi) = Med(\pi\pi) = Med(\pi) \cap Med(\pi) = Med(\pi)$. □ □ □

First we present a simple class of graphs, other than the cube-free median graphs, that have the intersecting-intervals property. Note that $K_{2,3}$ is not a median graph: the profile (u, v, w), consisting of the three vertices of degree 2, has the two vertices of degree 3 in the intersection $I(u, v) \cap I(v, w) \cap I(w, u)$. Clearly, it can also not be an induced subgraph of a median graph.

Theorem 6 *The complete bipartite graph $K_{2,n}$ has the intersecting-intervals property, for $n \geq 1$.*

Proof. For $n = 1, 2$ the graph $K_{2,n}$ is a cube-free median graph, so we are done (anyway, it is a simple exercise to prove that $K_{2,1}$ and $K_{2,2} = C_4$ have the property).

Let $n \geq 3$, let V be the vertex set of $K_{2,n}$, and let u, v be the vertices of degree 2. Let π be an even profile. If π contains u as well as v, then we can permute π such that it results in the profile $(u, v)\pi'$. Since $I(u, v) = V$, we need only to find a permutation of π' that produces the pairing for the intersecting intervals. Therefore we only need to consider even profiles that do not contain both u and v.

Without loss of generality, let π be an even profile that does not contain v. Let $G = K_{2,n} - v$ be the subgraph obtained by deleting vertex v. Then G is the star $K_{1,n}$. In particular, G is a cube-free median graph, whence it has the intersecting-intervals property. Note that we have $I_G(w, z) \subseteq I_{K_{2,n}}(w, z)$, for any two vertices w and z in G. There exists a permutation of π that results in the profile $(y_1, y_2, \ldots, y_m)$ such that

$$\bigcap_{i=1}^{m} I_G(y_{2i-1}, y_{2i}) \neq \emptyset.$$

Hence also $\bigcap_{i=1}^{m} I_{K_{2,n}}(y_{2i-1}, y_{2i})$ is nonempty. □ □ □

If we *glue together* two graphs along a vertex, then we mean that we take two graphs, select a vertex in each graph, and then construct a larger graph by identifying the two selected vertices.

Theorem 7 *Let G and H be two graphs having the intersecting-intervals property. If we glue G and H together along a vertex, then the resulting graph also has the intersecting-intervals property.*

Proof. Let x be the vertex along which we glue G and H together, and let Q be the resulting graph. Let u, v be two vertices in Q. If u, v are in G, then we have $I_Q(u, v) = I_G(u, v)$. If u, v are in H, then we have $I_Q(u, v) = I_H(u, v)$. And if u is in G and v in H, then we have $I_Q(u, v) = I_G(u, x) \cup I_H(x, v)$.

Let π be an even profile on Q, let π_G be the subprofile of elements in $Q - H$, let π_H be the subprofile of elements in $Q - G$, and let π_x be the subprofile of elements equal to x. Without loss of generality, we may assume that $|\pi_G| \geq |\pi_H|$. We consider two cases.

Case 1. $|\pi_G| = |\pi_H|$.
Note that in this case $|\pi_x|$ is even. Write $\pi_G = (a_1, a_2, \ldots, a_\ell)$ and $\pi_H = (b_1, b_2, \ldots, b_\ell)$. Clearly, for each pair (a_j, b_j), the interval $I(a_j, b_j)$ contains x. So x is contained in $\bigcap_{j=1}^{\ell} I(a_j, b_j)$. Set $\pi_1 = (a_1, b_1)(a_2, b_2) \ldots (a_\ell, b_\ell)$. Finally, π_x being even, we can pair off the copies of x in π_x. For each such pair we have $I(x, x) = \{x\}$. This pairing together with the pairing in π' produces a pairing of the elements of π such that the associated intervals have a non-empty intersection.

Case 2. $|\pi_G| > |\pi_H|$.
We construct the profile π'_H by replacing all elements in π_H by x. In doing so, we remember from which element in π_H each of these x's came. Let $\rho = \pi_G \pi_x \pi'_H$. Then ρ is an even profile on G. So there is a pairing of the elements of ρ such that the associated intervals have non-empty intersection. We consider two subcases.

Subcase (i). There is no pair with both elements in π'_H.
Note that each element of π'_H is paired with an element in $\pi_G \pi_x$. Now we recall from which element in π_H any x in π'_H came and replace it by this element. Then we have a pairing of the elements of π. Each associated interval contains the respective interval in the pairing of ρ. So the intervals of the pairing of π still have non-empty intersection.

Subcase (ii). There is a pair with both elements in π'_H.
Let x_u and x_v be a pair with x_u and x_v both in π'_H, where x_u came from u in $Q - G$ and x_v came from v in $Q - G$. So one of the intervals of the pairing is $I(x_u, x_v) = I(x, x) = \{x\}$. Hence the intersection of the intervals is exactly $\{x\}$, so x is in all the intervals.

Since $|\pi_G \pi_x| = |\pi_G| + |\pi_x| \geq |\pi_G| > |\pi_H|$, there has to be a pair (a, b) with both a and b in $\pi_G \pi_x$. We take the pairs (a, u) and (b, v) in π. Both intervals $I(a, u)$ and $I(b, v)$ in Q contain x. Now we delete the elements a, b, u, v from π, and the elements

a, b, x_u, x_v from ρ. Note that we delete two elements from $\pi_G \pi_x$ and two elements from π_H as well as two elements from π'_H. So in the remaining profiles there is still a surplus in G. Therefore the remaining profiles are still of the type of Case 2. Consider the remaining pairs of ρ. If there is a pair with both elements in π'_H, then we repeat this process. In the resulting profiles there is a surplus in G. We continue until there is no pair left with both elements in π'_H. But then we have a profile of the type in Subcase (i), which gives us a pairing of the remaining elements of π. All the way, we keep constructing pairs of elements of π, of which the interval contains x. Hence we are done. □ □ □

With the cube-free median graphs and the complete bipartite graphs $K_{2,n}$ as building stones we can now construct a non-trivial class of graphs that all have the intersecting-intervals property, and hence, by Theorem 5, have *Med* as unique *ABC*-function.

5 Weekly Ordered Coalitions

In this section we study a certain type of *ABC*-functions for voting on n alternatives. First we give a motivation for functions of this type. The Electorial Council has to decide what rules to impose on the voting process. The aspiration is to come up with a voting procedure that is as fair as possible, and also satisfies some basic rules. But the Council has problems to come up with the rules. On many issues no decision can be made. One thing is clear. If a majority comes up, then this will be the outcome of the voting. This implies that on K_n axioms (B) and (F) are two of the rules that are accepted unanimously by the Electorial Council. Moreover, the Electorial Council, being at least decent, also decides unanimously that (A) and (C) are always a rule. Hence the voting procedure, or consensus function, will always satisfy (A), (B) and (C). But now the situation becomes a little fuzzy, depending on the composition of the set of candidates, that is, the vertex set of K_n. Although fairness is being aspired, some things can happen that might disturb this. One thing is that a group of candidates might form a winning coalition, by which they can mess up the possibilities of the other candidates being elected (unless, of course, one or two of these get a majority of the votes). Another complication is that some candidates might have prevalence over others, which basically amounts to a weak order on the set of candidates. And then again, both phenomena might happen at the same time: there is a coalition that is weakly ordered. A certain level of fairness is still the goal. So the outcome should always consist of plurality vertices in some sense. Only those vertices are counted in the plurality count that belong to the winning coalition, and amongst these the highest ranked vertices in the weak order will be the winning vertices. From the title of this section we deduce that the weak order of the coalition is updated every week.

Let us formalize this. A *weak order* $\preceq$ on a set S is a relation on S such that there exists a partition $S_1, S_2, \ldots, S_p$ of S with $u \preceq v$ if and only if there exist $i \leq j$ with $u \in S_i$ and $v \in S_j$. Note that, for two vertices u and v in S_i, we have $u \preceq v$ as well as $v \preceq u$. If $i < j$, then we write $u \prec v$. Loosely speaking, a weak order is a linear order of the parts of the partition, or an *ordered partition*. We say that we *prefer* v over u if $u \prec v$. We are *indifferent* if $u \preceq v$ as well as $v \preceq u$, that is, u and v belong to the same part of the partition. The *indifference order* on S is the weak order with just the one part S in the partition. So, restricted to any one part of the partition, $\preceq$ is just the indifference order. A *linear order* is a weak order where every part of the partition is a singleton. For a subset S' of S, we say that v in S' has *highest preference* in S' if $w \preceq v$ for all $w \in S'$. Let K_n be the complete graph on n vertices with vertex set V. We say that $\preceq$ is a weak order on K_n if it is a weak order of the vertices of K_n. A fixed nonempty subset S of V is called a *winning coalition*, or *coalition* for short. Let $\preceq$ be a weak order on S. Let π be a non-tie, non-majority profile. An *S-plurality vertex* of π is a vertex in π from S that has highest occurrence amongst the vertices in S. Here a somewhat strange phenomenon might occur: π does not contain vertices from S. But then simply all vertices in S are S-plurality vertices (all occurring zero times). The *S-plurality set* of π is the set of all S-plurality vertices of π. Note that this set is nonempty. Also note that the vertices in $V - S$ are being completely ignored in this count. By T_π we denote the S-plurality vertices of π with highest rank in the weak order $\preceq$. If π does not contain any vertices from S, then T_π is just the set of vertices of highest rank with respect to $\preceq$ in S, that is, the part with highest preference.

Now we present our first example of a non-trivial family of ABC-functions on K_n. We define the *weakly ordered coalition function* $L_{S,\preceq} : V^* \to 2^V$ with coalition S and weak order $\preceq$ on S as follows.

$L_{S,\preceq}(\pi) = \{x\}$, for profiles π with a unique majority vertex x,
$L_{S,\preceq}(\pi) = \{x, y\}$, for tie profiles π on x, y,
$L_{S,\preceq}(\pi) = T_\pi$, for any non-tie, non-majority profile π.

Theorem 8 *Let S be a coalition in K_n, and let $\preceq$ be a weak order on S. Then the weakly ordered coalition function $L_{S,\preceq}$ on K_n satisfies (A), (B) and (C).*

Proof. For convenience we write $L = L_{S,\preceq}$. It is clear from the definition of L that L satisfies (A) and (B). So we only have to prove that L is consistent. Let ρ and σ be profiles with $L(\rho) \cap L(\sigma) \neq \emptyset$.

If ρ and σ are both a tie profile or a majority profile, then, by Lemma 1, we are done. So we may assume without loss of generality that ρ is a non-tie, non-majority profile. In this case $L(\rho) = T_\rho$. We consider three cases depending on the type of profile that σ is. We write $\pi = \rho\sigma$.

Case 1. σ is a tie profile on x, y.
In this case we have $T_\rho \cap \{x, y\} \neq \emptyset$. If $\{x, y\} \subseteq T_\rho$, then we have $T_\pi = \{x, y\}$, and we are done. So assume that $T_\rho \cap \{x, y\} = \{x\}$. If y is not in S, then $T_\pi = \{x\}$, and we are done. So assume that y is in S as well. Now we have to look at the number of occurrences of y in π. Since x is an S-plurality vertex of ρ, it follows that y cannot occur more often than x in ρ. If y occurs less, then, obviously, $T_\pi = \{x\}$, and we are done. If y occurs equally often as x in π, then we must have $y \prec x$, since y is not in T_ρ, and again we have $T_\pi = \{x\}$. This settles Case 1.

Case 2. σ is a majority profile with majority vertex x.

Now x lies in T_ρ. So, among the vertices of S, it has highest occurrence in ρ. So x is the single vertex in S with highest occurrence in π, that is, $T_\pi = \{x\}$, and we are done with this case.

Case 3. σ is a non-tie, non-majority profile.
Now we have $L(\sigma) = T_\sigma$, and $T_\rho \cap T_\sigma \neq \emptyset$. Each vertex in ρ occurs less than half of the times in ρ, and each vertex in σ occurs less than half of the times in σ, so none of the vertices in $\{\rho\} \cup \{\sigma\}$ occurs at least half of the times in $\pi = \rho\sigma$. So π is non-tie as well as non-majority. Hence $L(\pi) = T_\pi$. Clearly, the vertices from S with highest occurrence in π cannot have higher rank than those in $\{\rho\} \cup \{\sigma\}$. So we have $T_\pi = T_\rho \cap T_\sigma$, which settles Case 3, and completes the proof. □ □ □

The two extreme cases occur when S is a single vertex, and when $S = V$. We consider the latter case in more detail in the next section. For the former case, we fix a vertex p in K_n, and call it the *fixt point*. We define the *fixt point function* L_p with fixt point p as follows:

$L_p(\pi) = \{x, y\}$, for any tie profile π on x, y.
$L_p(\pi) = \{x\}$, for any majority profile π with majority vertex x.
$L_p(\pi) = \{p\}$, for any other profile π, so non-tie and non-majority.

This function is precisely the weakly ordered coalition function with winning coalition $S = \{p\}$. Hence it is an ABC-function. Of course, it satisfies the axioms that necessarily follow from (A), (B) and (C). For $n = 3$, it also satisfies $(Supp)$, but none of the other axioms in Section 3. For $n \geq 4$, it does not satisfy any of the other axioms in Section 3. Clearly, there are additional axioms needed for an axiomatic characterization. We leave this here as an open problem.

6 Weak Order Functions

In this section we study the weakly ordered coalition functions where V is the winning coalition.

Let $\preceq$ be a weak order on V. The *weak order function* $L_\preceq$ of the weak order $\preceq$ is the function defined as follows:

$L_\preceq(\pi) = \{x, y\}$ if π is a tie-profile on x, y.
$L_\preceq(\pi) = \{x \mid x$ is a plurality vertex of π with highest preference w.r.t. $\preceq\}$.

Note that, if we take $\preceq$ to be the indifference order on K_n, then we get the median function, that is, in this case $L_\preceq = Med$.

Theorem 9 *Let $\preceq$ be a weak order on K_n. Then the weak order function $L_\preceq$ satisfies (A), (B), (C) and $(Plur)$*

Proof. Clearly, we have $L_\preceq = L_{V,\preceq}$ with V as the coalition. So, by Theorem 8, $L_\preceq$ satisfies (A), (B) and (C). That $L_\preceq$ satisfies $(Plur)$ follows immediately from its definition. □ □ □

In general the converse of this theorem is not true. In Section 7.1 we exhibit examples of consensus functions that satisfy (A), (B), (C) and $(Plur)$ that are not weak order functions, for $n \geq 4$. But in the case that $n = 3$ the converse is true, as we will see in Theorem 11.

Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile on K_n. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of K_n. The type of the profile is the n-tuple $[\ell_1, \ldots, \ell_n]$ with ℓ_i being the number of occurrences of v_i in π, for $i = 1, \ldots, n$. In the case of K_3, we write the vertex set as $V = \{u, v, w\}$ with $u = v_1$, $v = v_2$ and $w = v_3$. Now we write the type of a profile π as $[i, j, k]$, where i is the number of occurrences of u, and j that of v, and k that of w. If L satisfies (A), then profiles of the same type have the same output. If π is of type $[i, j, k]$, then we write $L(\pi) = L[i, j, k]$.

Our next theorem is the special case of Theorem 14 below, viz. for K_3. It singles out the median function in the family of all ABC-functions on K_3. We state it here already as a separate result, because it is a case in the proof of Theorem 11.

Theorem 10 *Let L be a consensus function on K_3 with vertex set $\{u, v, w\}$. Then $L = Med$ if and only if L satisfies (A), (B), (C) and $(Full)$.*

Proof. If $L = Med$, then L satisfies the four axioms.

For the converse, recall that the axiom $(Full)$ says: $L(u, v, w) = \{u, v, w\}$. Because $L(\pi) = Med(\pi)$ for all profiles π with $|\{\pi\}| \leq 2$ on any graph, we only need to consider profiles of the type $[i, j, k]$ with, say, $1 \leq i \leq j \leq k$. For $i = j = k$ we are done by (C) and the fact that $L(u, v, w) = \{u, v, w\} = Med(u, v, w)$. So let $i < k$.

First suppose that $i = j < k$. Then $L[i, i, i] \cap L[0, 0, k - i] = \{w\} = Med(i, j, k)$. By (C), we have $L[i, j, k] = Med[i, j, k]$. Next suppose that $i < j \leq k$. Now

$$L[i, i, i] \cap L[0, j - i, k - i] = L[0, j - i, k - i] =$$

$$Med[0, j - i, k - i] = Med[i, j, k].$$

And again, by (C), we are done. □ □ □

Now we present an example of a family of ABC-functions that can be characterized by an additional axiom, viz. $(Plur)$. This was one of our goals for the ABC-Problem.

Theorem 11 *Let L be a consensus function on K_3 with vertex set $\{u, v, w\}$. Then L satisfies (A), (B), (C) and $(Plur)$ if and only if there exists a weak order $\preceq$ on K_3 such that $L = L_{\preceq}$.*

Proof. By Theorem 9 we only need to prove the only if part. So let L satisfy (A), (B), (C) and $(Plur)$. Set $V = \{u, v, w\}$.

Since L satisfies $(Plur)$, we know that $L(\pi) \subseteq Pl(\pi) = Med(\pi)$, for any profile π. By Theorem 10, we can take the indifference order in the case that $L(u, v, w) = \{u, v, w\}$.

So let $L(u, v, w)$ be a proper subset of V. Now we have to find the partition and weak order on V. It turns out that we have to be careful here, because of the fact that, as soon as the carrier set of a profile consists of one or two vertices the weak order is not involved in the definition of L, that is, we cannot use $L(u,v)$ to define a part of the partition, see below. Due to (A), we only need to consider two cases.

Case 1. $L(u,v,w)=\{v,w\}$.
Let m be a positive integer. By (C) and (A), we have $L[m,m,m]=\{v,w\}$. Define the weak order $\preceq$ by the partition $V_1=\{u\}$ and $V_2=\{v,w\}$. We only need to consider profiles of the type $[i,j,k]$ with $i,j,k\geq 1$. If $i<\max(j,k)$, then $u\notin Med[i,j,k]$, so $u\notin L[i,j,k]$. Set $m=\min(i,j,k)$. Then $[i-m,j-m,k-m]$ is a profile with carrier set containing one or two vertices and $i-m<\max(j-m,k-m)$. So $L[i-m,j-m,k-m]=Med[i-m,j-m,k-m]$ does not contain u. Hence we have $L[i-m,j-m,k-m]\cap L[m,m,m]\neq\emptyset$. So, by (C), we have

$$L[i,j,k]=L[i-m,j-m,k-m]\cap L[m,m,m]=$$

$$L[i-m,j-m,k-m]\cap L[u,v,w]=Med[i-m,j-m,k-m]\cap\{v,w\}.$$

If $j=k$, then it follows that $L[i,j,k]=\{v,w\}$. Otherwise, we may assume that $j<k$. Then $L[i,j,k]=\{w\}$. In both cases $L[i,j,k]$ consists of the vertices with highest preference amongst the vertices with highest occurrence.

Now suppose that $i=\max(j,k)$. If $j=k$ we are done. So we may take $j<k$, whence $i=k$. Consider the profile $[i-j,0,k-j]$. Then

$$L[i-j,0,k-j]=Med[i-j,0,k-j]=\{u,w\}.$$

So $L[j,j,j]\cap L[i-j,0,k-i]=\{v,w\}\cap\{u,w\}=\{w\}$. By Consistency we have $L[i,j,k]=\{w\}$. Now in $[i,j,k]$ the vertices u and w occur most often, and amongst these w has highest preference. Hence we are done.

Finally suppose that $i>\max(j,k)$. Now we have $L[i,j,k]\subseteq Med[i,j,k]=\{u\}$. So $L[i,j,k]=\{u\}$, and again we are done.

Case 2. $L(u,v,w)=\{w\}$.
In the sequel we will consider weak orders $\preceq$ on V in which $u,v\prec w$. For the time being we do not yet consider how u and v compare in the weak order. Consider any profile $[i,j,k]$ with $i,j\leq k$. We have to prove that $L[i,j,k]=\{w\}$. If $i=j=k$, then we are done by (C). So suppose that $\min(i,j)<k$. Assume that $i\leq j$ (otherwise we reverse the role of i and j in the following argument). So $i<k$. Consider the profile $[0,j-i,k-i]$. Then

$$L[0,j-i,k-i]\cap L[i,i,i]=Med[0,j-i,k-i]\cap\{w\}=\{w\}.$$

Hence by (C), we have $L[i,j,k]=\{w\}$. Now w is the vertex with highest occurrence in $[i,j,k]$ and it has higher preference than any other vertex. So we are done.

For profiles $[i,j,k]$ with $k<\max(i,j)$ we need to know how u and v are ordered with respect to $\preceq$. To determine $\preceq$ on $\{u,v\}$ we proceed as follows.

Note that now we cannot determine $L(u,u,v,v,w)$ by using (C) and the subprofiles (u,v,w) and (u,v), because the outputs with respect to L of these two profiles have empty intersection. Note that $L(u,u,v,v,w)\subseteq Med(u,u,v,v,w)=\{u,v\}$. Due to (A), we only need to consider two subcases.

Subcase 2.1. $L(u,u,v,v,w) = \{u,v\}$.
We define the partition $V_1 = \{u,v\}$ and $V_2 = \{w\}$. Let $\preceq$ be the weak order of this partition.

First we prove that for profiles $[i,i,k]$ with $i > k$ we have $L[i,i,k] = \{u,v\}$. The first step is that by (C) we have $L[2k,2k,k] = \{u,v\}$. Consider the profile $[k{+}p,k{+}p,k]$ with $0 < p < k$. Note that

$$L[k+p,k+p,k] \subseteq Med[k+p,k+p,k] = \{u,v\}.$$

So $L[k-p,k-p,0] \cap L[k+p,k+p,k]$ is nonempty. Hence

$$\{u,v\} = L[2k,2k,k] = L[k-p,k-p,0] \cap L[k+p,k+p,k],$$

which implies that $L[k+p,k+p,k] = \{u,v\}$. Consider the profile $[k+q,k+q,k]$ with $q > k$. Then $L[2k,2k,k] \cap L[q-k,q-k,0] = \{u,v\}$. So $L[k+q,k+q,k] = \{u,v\}$.

Next consider any profile $[i,j,k]$ with $k < \max(i,j)$ and $i \neq j$. Now either $i > j,k$ of $j > i,k$. in the first case $L[i,j,k] \subseteq Med[i,j,k] = \{u\}$, and we are done. The latter case follows similarly.

Subcase 2.2. $L(u,u,v,v,w) = \{v\}$.
We define the partition $V_1 = \{u\}$, $V_2 = \{v\}$, and $V_3 = \{w\}$. Let $\preceq$ be the weak order of this partition. Recall that $k < \max(i,j)$. If either $i > j$ or $j > i$ then $Med[i,j,k]$ consists of a single vertex and we are done by the inclusion property $L(\pi) \subseteq Med(\pi)$. So we only need to consider the profiles $[i,i,k]$ with $i > k$. By (C), we have $L[2k,2k,k] = \{v\}$. Consider the profile $[k+p,k+p,k]$ with $0 < p < k$. Note that

$$L[k+p,k+p,k] \subseteq Med[k+p,k+p,k] = \{u,v\}.$$

So $L[k-p,k-p,0] \cap L[k+p,k+p,k]$ is nonempty. Hence

$$\{v\} = L[2k,2k,k] = L[k-p,k-p,0] \cap L[k+p,k+p,k],$$

which implies that $L[k+p,k+p,k] = \{v\}$. Consider the profile $[k+q,k+q,k]$ with $q > k$. Then $L[2k,2k,k] \cap L[q-k,q-k,0] = \{v\}$. So $L[k+q,k+q,k] = \{v\}$. □ □ □

Note that, in Section 7.1, we will see that the four axioms in Theorem 11 do not characterize the weak order functions on K_n with $n \geq 4$.

We close this section with some observations.

Lemma 12 *Let L be a consensus function on K_n satisfying (A), (B) and (C). Then L satisfies $(Plur)$ if and only if L satisfies (Pl_1).*

Proof. Trivially $(Plur)$ implies (Pl_1) on any graph.

To prove the converse, let π be a profile with at least two plurality vertices, and let v and w be two plurality vertices. Assume that $L(\pi)$ contains a vertex that is not a plurality vertex, say u. Then, by (B), we have that u is in $L(\pi) \cap \{u,v\} = L(\pi) \cap L(u,v)$. Let $\rho = \pi(u,v)$ be the concatenation of π and (u,v). Then, by (A) and (C), u is in $L(\pi(u,v))$. But in the profile $\pi(u,v)$ vertex v is the unique plurality vertex. So, by (Pl_1) we have $L(\pi(u,v)) = \{v\}$. This impossibility completes the proof □ □ □

Thus we get another characterization of the weak order functions on K_3.

Proposition 13 *Let L be a consensus function on K_3. Then L satisfies (A), (B), (C) and (Pl_1) if and only if there exists a weak order $\preceq$ on K_3 such that $L = L_{\preceq}$.*

6.1 The Median Function

On K_n the median function Med is just the plurality function Pl. Already as early as 1991, Roberts [27] gave various axiomatic characterizations of Pl on K_n. One of his sets of axioms was (A), (C), (F) and the so-called neutrality axiom. For a definition of this axiom, and for other characterizations, we refer to his paper [27]. Here we present another characterization that fits into our ABC-program, although (B) does not appear explicitly in the list of axioms. We use axiom (Uni), which implies (B) and $(Full)$ on the complete graph, and (F) on any connected graph. Note that with Theorem 10 above we already had a characterization of Med on K_3.

Theorem 14 *Let L be a consensus function on K_n. Then L satisfies (A), (C), (Uni) if and only if $L = Med$.*

Proof. Write π, using (A), in its notation $[\ell_1, \ldots, \ell_n]$ with $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. Let m be the minimum of the values of the positive ℓ_i's. Then we split off the profile $[m, m, \ldots, m, \ldots, 0]$, where the number of m's is precisely the number of non-zero ℓ_i's, say p. By (C) and (Uni) we have $L[m, m, \ldots, m, 0, \ldots, 0] = \{v_1, v_2, \ldots, v_p\}$. We repeat this process until we arrive at a subprofile π' that is a uniform profile. Note that π' contains precisely the plurality vertices of π. Then the L-outputs of these subprofiles all contain $\{\pi'\}$, whence, by consistency, $L(\pi) = L(\pi') = Med(\pi)$. □ □ □

6.2 Linear Order Function on K_3

The median function is the weak order function with the indifference order as weak order. In this section we consider the linear order on K_n, that is, every part in the weak order is a singleton. Again we have a complete axiomatic characterization on K_3. We leave the case $n \geq 4$ as an open problem. We need the following axiom.

$(V1)$: $|L(\pi)| = 1$ for all π with $\{\pi\} = V$.

Theorem 15 *Let L be a consensus function on K_3. Then L is a linear order function if and only if L satisfies (A), (B), (C), $(Plur)$ and $(V1)$.*

Proof. Since the only if direction is obvious, we let L be a consensus function on K_3 satisfying (A), (B), (C), $(Plur)$ and $(V1)$. Denote the vertex set of K_3 by $\{u, v, w\}$. Without loss of generality, assume that

(i) $L(u, v, w) = \{w\}$.

(ii) $L(u, u, v, v, w) = \{v\}$.

Let $\leq$ be the linear order $u \leq v \leq w$. We will show that $L = L_{\leq}$.

Let π be a profile on K_3. If $|\{\pi\}| \leq 2$, then $(Plur)$ and (B) imply that $L(\pi) = L_{\leq}(\pi)$. So assume $\{\pi\} = \{u, v, w\}$. If π has a unique plurality vertex, say x, then $L(\pi) = \{x\} = L_{\leq}(\pi)$. If π is uniform then assumption (i) above together with (C) implies that $L(\pi) = \{w\} = L_{\leq}(\pi)$. So we assume that the plurality set of π contains precisely two vertices.

Case 1. $Pl(\pi) = \{v, w\}$.
After possibly permuting elements, π is the concatenation of some positive number α of copies of (u, v, w) and some positive number $\beta > 0$ of copies of (v, w). Then by (i), (B) and (C) it follows that $L(\pi) = \{w\} = L_{\leq}(\pi)$.

Case 2. $Pl(\pi) = \{u, w\}$.
Argue as in Case 1 with v replaced by u.

Case 3. $Pl(\pi) = \{u, v\}$.
After possibly permuting elements, π is the concatenation of some positive number α of copies of (u, v) and some positive number β of copies of (u, v, w). If $\alpha = \beta$ then π is (a possible reordering of) α copies of (u, u, v, v, w). Hence (C) together with our assumption (ii) implies that $L(\pi) = \{v\} = L_{\leq}(\pi)$.

If $\alpha > \beta$ then (after rearranging) π is the concatenation of $\alpha - \beta$ copies of (u, v) and β copies of (u, u, v, v, w). Hence condition (ii) and (B) and (C) imply $L(\pi) = \{v\} = L_{\leq}(\pi)$.

So assume $\alpha < \beta$. We want to show that $L(\pi) = L_{\leq}(\pi) = \{v\}$. Assume the contrary. Then, by $(Plur)$, we have u in $L(\pi)$. Let ρ be the profile obtained by concatenating $\beta - \alpha$ copies of (u, v) with π. Then u is in $L(\pi) \cap L(u, v)$. So, by (C), we have

$$L(\rho) = L(\pi) \cap L(u, v).$$

But, after possible rearranging, ρ is the concatenation of β copies of (u, u, v, v, w). Hence, by (ii) and (C), we have $L(\rho) = \{v\}$. This contradiction concludes the proof.
□ □ □

7 Partisan voting on K_n

Here we present two partisan voting procedures on K_n that satisfy (A), (B) and (C). On K_3 these procedures turn out to be instances of weakly ordered coalition functions, but in the case of K_n, with $n \geq 4$, we get two new families of ABC-functions. The family in Section 7.1 also satisfies $(Plur)$, but, for $n \geq 4$, there are many more functions in the family than the weak order functions. The family in Section 7.2 satisfies (A), (B) and (C), and their immediate consequences (F) and (U), but none of the other axioms in Section 3, for $n \geq 4$.

7.1 Partisan voting without party leaders

We take a partition $\Gamma = \{F_1, F_2, \ldots, F_p\}$ of V into p nonempty *parties* F_1 up to F_p. Using the partition Γ we define the ABC-function L_Γ that represents a partisan voting involving the parties in Γ. In the voting process we assume always that members of the same party vote for their own party, and their votes are counted as *party votes*. Let π be a profile. So, as usual, we have $L_\Gamma(\pi) = M_\pi$, if $M_\pi \neq \emptyset$, in which case π is either a tie-profile or a majority profile. If π is non-tie, non-majority, then the output $L_\Gamma(\pi)$ will be a subset of the plurality vertices of π. In picking the vertices among the plurality vertices we proceed as follows. We count the votes for each party, and the plurality vertices that have the highest number of party votes are chosen as output vertices. Note that a party can have more than one member elected.

It follows from the behaviour of L on profiles with majority vertices that L_Γ satisfies (A) and (B). Also L_Γ satisfies $(Plur)$ trivially.

Theorem 16 *Let V be the vertex set of K_n, and let $\Gamma = \{F_1, F_2, \ldots, F_p\}$ be a partition of V into p parties. Then L_Γ satisfies (C).*

Proof. Take two profiles π and ρ with $L_\Gamma(\pi) \cap L_\Gamma(\rho) \neq \emptyset$. We consider three cases.

Case 1. $M_\pi \neq \emptyset$ and $M_\rho \neq \emptyset$.
This case follows immediately from Lemma 1.

Case 2. $M_\pi \neq \emptyset$ and $M_\rho = \emptyset$.
Note that the case $M_\pi = \emptyset$ and $M_\rho \neq \emptyset$ follows similarly.

Now $L_\Gamma(\pi) = M_\pi$. If $M_\pi = \{x\}$, then x lies in $L_\Gamma(\rho)$. So it is a plurality vertex of ρ. But then $\pi\rho$ has x as single plurality vertex, whence $L_\Gamma(\pi\rho) = \{x\}$. If $M_\pi = \{x, y\}$, then π is a tie-profile, and we can write π as the concatenation $(x, y)^t$ of t copies of (x, y). Now $L_\Gamma(\rho) \cap \{x, y\} \neq \emptyset$. Since $L_\Gamma(\rho) \subseteq Pl(\rho)$, we have that $Pl(\pi\rho) \subseteq \{x, y\}$. So $L_\Gamma(\pi\rho) \subseteq \{x, y\}$. The number of party votes on x or y in $\pi\rho$ is the number party votes in ρ plus t. So the number of party votes in ρ determines whether x and/or y are in $L_\Gamma(\pi\rho)$. So we have $L_\Gamma(\pi\rho) = L_\Gamma(\pi) \cap L_\Gamma(\rho)$, and we are done.

Case 3. $M_\pi = M_\rho = \emptyset$.
In this case we get the plurality vertices with the highest number of party votes as output for each of the profiles π and ρ. Write $\pi_1 = \pi$ and $\pi_2 = \rho$. For $i = 1, 2$ let z_i be a vertex in $L_\Gamma(\pi_i)$. Let t_i be the number of occurrences of z_i in π_i, so that no vertex occurs more than t_i times in π_i. Let s_i be the number of other party votes that belong to the same party as z_i. So any other plurality vertex of π_i has not more than s_i other party members in π_i. Now let z be any vertex in $L_\Gamma(\pi_1) \cap L_\Gamma(\pi_2)$. Then z occurs $t_1 + t_2$ times in $\pi_1\pi_2$, and no vertex occurs more often. Moreover z has $s_1 + s_2$ other party members in $\pi_1\pi_2$. Let y be any vertex not in this intersection. If y is not in $Pl(\pi_1) \cap Pl(\pi_2)$, then it is not a plurality vertex of $\pi_1\pi_2$, so it is not in $L_\Gamma(\pi_1\pi_2)$. If y is a plurality vertex of both π_1 and π_2, then it has at most s_1 other party members in π_1 and at most s_2 other party members in π_2 but no equality on both cases. So it has less than $s_1 + s_2$ other party members in $\pi_1\pi_2$. From these observations we deduce that $L_\Gamma(\pi_1\pi_2) = L_\Gamma(\pi_1) \cap L_\Gamma(\pi_2)$. This concludes the proof of consistency. □ □ □

If there is only one party, or if all parties are singletons, then it is just the median function. The median function seems to pop up almost everywhere. So it is indeed an important function, even on K_n. Note that L_Γ satisfies $(Plur)$. Therefore, on K_3 it is just a weak order function. But on K_n with $n \geq 4$ there are many other functions of type L_Γ. So this example washes any conjecture down the drain about weak order functions being the $(Plur)$ functions on K_n. Question: can we find extra axioms that will give us the weak order functions on K_n with $n \geq 4$?

7.2 Partisan voting with party leaders

There is another type of partisan voting that yields a family of ABC-functions. Now a *party* (D, q) consists of a nonempty subset D of V and a special vertex q in D, called its *leader*. A *party partition* is a partition Π^* of V, where each part is a party with a party leader. Let $(D_1, q_1), (D_2, q_2), \ldots, (D_r, q_r)$ be the parties in the party partition. The function L_{Π^*} on K_n with party partition Π^* is defined as follows:

$L_{\Pi^*}(\pi) = M_\pi$ in case M_π is nonempty.

For non-tie, non-majority profiles π, the votes for each party are aggregated. So the counts are per party. Now the parties that have the most votes (that is, the 'plurality parties') get their leaders in the output of $L_{\Pi^*}(\pi)$.

We can formulate this partisan voting with party leaders in another way. For non-tie, non-majority profiles we could also do the following thought experiment. We replace the vote of each party member by a vote for its party leader. Thus we get a profile π' of the same length as π on K_r, the complete graph of only the party leaders, and the output is now the median set of this profile π'. Let Med_{K_r} be the median function on K_r. Then this translation gives us

$$L_{\Pi^*}(\pi) = Med_{K_r}(\pi').$$

Trivially, this function satisfies (A) and (B).

Theorem 17 *Let Π^* be a party partition with party leaders of the vertex V of K_n and let L_{Π^*} be its partisan function. Then L_{Π^*} satisfies (C).*

Proof. Take two profiles π and ρ with $L_{\Pi^*}(\pi) \cap L_{\Pi^*}(\rho) \neq \emptyset$. We consider three cases.

Case 1. $M_\pi \neq \emptyset$ and $M_\rho \neq \emptyset$.
This case follows immediately from Lemma 1.

Case 2. $M_\pi \neq \emptyset$ and $M_\rho = \emptyset$.
Note that the case $M_\pi = \emptyset$ and $M_\rho \neq \emptyset$ follows similarly.

Now $L_{\Pi^*}(\pi) = M_\pi$. If $M_\pi = \{x\}$, then π is a majority profile with majority vertex x, and all other vertices occur less in π than x. Moreover x lies in $L_{\Pi^*}(\rho)$. So it is the party leader of a party with the largest number of votes in ρ. Clearly, this party is the unique party with the largest number of votes in $\pi\rho$. So, whether x is the unique majority vertex in $\pi\rho$ or not, we have $L_{\Pi^*}(\pi\rho) = \{x\}$.

If $M_\pi = \{x, y\}$, then π is a tie-profile $(x, y)^t$. Now $L_{\Pi^*}(\rho) \cap \{x, y\} \neq \emptyset$. If any of x and y is in $L_{\Pi^*}(\rho)$, then it is the party leader of a party with the largest number of votes in $\pi\rho$. Any other vertex cannot be such a vertex in $\pi\rho$. So again we have $L_{\Pi^*}(\pi\rho) = L_{\Pi^*}(\pi) \cap L_{\Pi^*}(\rho)$.

Case 3. $M_\pi = M_\rho = \emptyset$.
In this case we use the K_r representing the party leaders. We get the party leaders of the parties with the largest number of party votes as output for each of the profiles π and ρ. If x is a party leader of a party with the largest number of votes in both π and

ρ, then it is that of the concatenation $\pi\rho$ as well. And if it is not then it is neither of the concatenation. So again we have

$$L_{\Pi^*}(\pi\rho) = Med_{K_r}((\pi\rho)') = Med_{K_r}(\pi'\rho') =$$

$$Med_{K_r}(\pi') \cap Med_{K_r}(\rho') = L_{\Pi^*}(\pi) \cap L_{\Pi^*}(\rho),$$

which concludes the proof. □ □ □

Thus we get a whole new family of ABC-functions. Well, not completely new. If each party consists of a single vertex (which is then the leader), then $L_{\Pi^*} = Med$. If the party partition consists of one party, then L_Π is the fixt point function with the party leader as fixt point. If there are two parties, one of which a singleton, then again the function is a fixt point function with the party leader of the other party as fixt point. These observations imply that on K_3 the partisan functions are just special weakly ordered coalition functions (viz. the fixt point functions and the median function), so no new functions arise here. Note that, when $n > 3$ and there are at least two parties and fewer than n parties, then L_{Π^*} satisfies (A), (B) and (C), and hence also (F) and (U), but none of the other axioms in Section 3.

8 An infinite family of ABC-functions

So far we have exhibited only finitely many ABC-functions on K_n with $n \geq 3$. The aim of this section is to provide an infinite family on K_3.

Let L be an ABC-function on K_3. Then, due to (A), there are basically three possibilities for the value of $L(u, v, w)$, viz. $\{u, v, w\}$, $\{v, w\}$ and $\{w\}$. The first case turns out to be just the median function, see Theorem 10. We focus here on the second case. We have various instances of weakly ordered coalition functions with $L(u, v, w) = \{v, w\}$, one of which is the weak order function with weak order $\{u\} \preceq \{v, w\}$. But in this case there is also an infinite family.

Let m be a positive integer. We define the location function L_m on K_3 as follows:

$$u \in L[i,j,k] \;\Leftrightarrow\; \left(i \geq j + \frac{1}{m}k \;\text{ and }\; j \geq k\right) \text{ or } \left(i \geq k + \frac{1}{m}j \;\text{ and }\; k \geq j\right);$$

$$v \in L[i,j,k] \;\Leftrightarrow\; i \leq j + \frac{1}{m}k \;\text{ and }\; j \geq k;$$

$$w \in L[i,j,k] \;\Leftrightarrow\; i \leq k + \frac{1}{m}j \;\text{ and }\; k \geq j.$$

By definition, L_m satisfies (A). Observe that $L[1,1,0] = \{u, v\}$, $L[1,0,1] = \{u, w\}$, and $L[0,1,1] = \{v, w\}$. Moreover, $L[2,0,0] = \{u\}$, $L[0,2,0] = \{v\}$, and $L[0,0,2] = \{w\}$. So L satisfies (B).

Note that we have $L[1,1,1] = \{v, w\}$. Moreover, we have

$$L[i,j,k] = \{u, v, w\} \;\Leftrightarrow\; j = k \text{ and } i = \frac{m+1}{m}k.$$

So $L_{m_1} \neq L_{m_2}$, for any two integers $m_1, m_2 \geq 2$ with $m_1 \neq m_2$. Therefore, we have infinitely many functions of this type.

For $m = 1$, the function L is not consistent: we have

$$L_1[3,1,2] \cap L_1[3,2,1] = \{u\},$$

but

$$L_1[6,3,3] = \{u, v, w\}.$$

We will show that L_m satisfies (C), for $m \geq 2$.

Theorem 18 *The location function L_m on K_3 satisfies (A), (B) and (C), for $m \geq 2$.*

Proof. As observed above L satisfies (A) and (B).

We now want to show that L satisfies (C). Let π_1 be represented by $[i_1, j_1, k_1]$ and let π_2 be represented by $[i_2, j_2, k_2]$. Then

$$v \in L(\pi_1) \cap L(\pi_2)$$

if and only if the two integer 3-tuples satisfy the inequalities:

$$i \leq j + \frac{1}{m}k \quad \text{and} \quad j \geq k.$$

So, in case that $v \in L(\pi_1) \cap L(\pi_2)$, it follows that

$$(i_1 + i_2) \leq (j_1 + j_2) + \frac{1}{m}(k_1 + k_2) \quad \text{and} \quad (j_1 + j_2) \geq (k_1 + k_2)$$

and so

$$v \in L(\pi_1\pi_2).$$

Now assume that $v \in L(\pi_1\pi_2)$ and $v \notin L(\pi_2)$. We will show that $L(\pi_1) \cap L(\pi_2) = \emptyset$. From $v \in L(\pi_1\pi_2)$ we deduce

$$i_1 + i_2 \leq (j_1 + j_2) + \frac{1}{m}(k_1 + k_2) \quad \text{and} \quad j_1 + j_2 \geq k_1 + k_2,$$

and from $v \notin L(\pi_2)$ we deduce

$$i_2 > j_2 + \frac{1}{m}k_2 \quad \text{or} \quad j_2 < k_2.$$

Case 1. $i_2 > j_2 + \frac{1}{m}k_2$ and $j_2 \geq k_2$.
Then

$$i_2 > j_2 + \frac{1}{m}k_2 \geq k_2 + \frac{1}{m}j_2$$

and so $L(\pi_2) = \{u\}$. On the other hand,

$$i_1 < j_1 + \frac{1}{m}k_1 \quad \text{and} \quad j_1 \geq k_1$$

or

$$i_1 < k_1 + \frac{1}{m} j_1 \text{ and } k_1 > j_1.$$

In either case, $u \notin L(\pi_1)$. Thus we have $L(\pi_1) \cap L(\pi_2) = \emptyset$.

Case 2. $i_2 > j_2 + \frac{1}{2} k_2$ and $j_2 < k_2$.
Now $j_1 + j_2 \geq k_1 + k_2$ along with $j_2 < k_2$ implies that $j_1 > k_1$. Moreover,

$$i_1 \leq j_1 + \frac{1}{m} k_1 \text{ and } j_1 > k_1$$

implies that $L(\pi_1) = \{v\}$. Since $v \notin L(\pi_2)$ it follows that $L(\pi_1) \cap L(\pi_2) = \emptyset$.

Case 3. $i_2 \leq j_2 + \frac{1}{m} k_2$ and $j_2 < k_2$.
Notice that $j_2 < k_2$ implies $j_2 + \frac{1}{m} k_2 < k_2 + \frac{1}{m} j_2$ and so

$$i_2 < k_2 + \frac{1}{m} j_2.$$

Next, $i_2 < k_2 + \frac{1}{m} j_2$ and $k_2 > j_2$ implies that $L(\pi_2) = \{w\}$. As in Case 2, $j_1 + j_2 \geq k_1 + k_2$ along with $j_2 < k_2$ implies that $j_1 > k_1$ and so $w \notin L(\pi_1)$. Thus, $L(\pi_1) \cap L(\pi_2) = \emptyset$.

We can now say that if $L(\pi_1) \cap L(\pi_2) \neq \emptyset$, then

$$v \in L(\pi_1) \cap L(\pi_2) \iff v \in L(\pi_1 \pi_2).$$

Since the roles of v and w are symmetric we can also say that if $L(\pi_1) \cap L(\pi_2) \neq \emptyset$, then

$$w \in L(\pi_1) \cap L(\pi_2) \iff w \in L(\pi_1 \pi_2).$$

Suppose $u \in L(\pi_1) \cap L(\pi_2)$, Then

$$i_1 \geq j_1 + \frac{1}{m} k_1 \text{ and } i_1 \geq k_1 + \frac{1}{m} j_1$$

and

$$i_2 \geq j_2 + \frac{1}{m} k_2 \text{ and } i_2 \geq k_2 + \frac{1}{m} j_2.$$

It follows that

$$i_1 + i_2 \geq (j_1 + j_2) + \frac{1}{m}(k_1 + k_2) \text{ and } i_1 + i_2 \geq (k_1 + k_2) + \frac{1}{m}(j_1 + j_2)$$

and so $u \in L(\pi_1 \pi_2)$.

Now assume $u \in L(\pi_1 \pi_2)$ and $u \notin L(\pi_2)$. We will show that $L(\pi_1) \cap L(\pi_2) = \emptyset$. The assumption $u \notin L(\pi_2)$ leads, up to symmetry, to two cases.

Case (i)**.** $i_2 < j_2 + \frac{1}{m} k_2$ and $i_2 < k_2 + \frac{1}{m} j_2$.
Since $u \in L(\pi_1 \pi_2)$ it follows that

$$i_1 + i_2 \geq (j_1 + j_2) + \frac{1}{m}(k_1 + k_2) \text{ and } i_1 + i_2 \geq (k_1 + k_2) + \frac{1}{m}(j_1 + j_2).$$

Therefore,

$$i_1 > j_1 + \frac{1}{m}k_1 \ \text{ and } \ i_1 > k_1 + \frac{1}{m}j_1.$$

This implies that $L(\pi_1) = \{u_1\}$. Since $u \notin L(\pi_2)$ it follows that $L(\pi_1) \cap L(\pi_2) = \emptyset$.

Case (ii)**.** $i_2 < j_2 + \frac{1}{2}k_2$ and $i_2 \geq k_2 + \frac{1}{2}j_2$ and $j_2 > k_2$.
It follows that $L(\pi_2) = \{v,\}$. Also,

$$i_1 > j_1 + \frac{1}{m}k_1$$

which implies that $v \notin L(\pi_1)$. Thus, $L(\pi_1) \cap L(\pi_2) = \emptyset$. This settles Case (ii).

So we have proved the following: if $L(\pi_1) \cap L(\pi_2) \neq \emptyset$, then

$$u \in L(\pi_1) \cap L(\pi_2) \Leftrightarrow u \in L(\pi_1\pi_2).$$

Hence we conclude that L satisfies (C), by which the proof is complete. □ □ □

9 Concluding Remarks

We have given some first answers for the ABC-Problem. Besides the median graphs we have another class on which the median function is the unique ABC-function, viz. the complete bipartite graphs $K_{2,n}$ with cube-free median graphs appended at its vertices of degree n. On K_3 the situation is different: nice families of ABC-functions are abundant, and we also have an infinite family. Most probably we are far from a complete classification of all ABC-functions on K_3. For some families we have nice axiomatic characterizations, such as the weak order functions and the linear order functions. Various families extend naturally to K_n with $n \geq 3$.

As could be expected, our first answers generate new questions. To name just one: is there a class of graphs on which there are more ABC-functions than just the median function, but only finitely many more?

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